Qualitative theory of spectrum, scattering, and decay control (Lessons on quantum intuition)

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Algorithms for constructing quantum systems with desired spectral parameters and scattering characteristics are discussed in this review. Instructive illustrations and simple qualitative explanations of them are used to show how the potential must be changed in order to eliminate a given level from the discrete spectrum without affecting the others or to create a new level at a given location, to shift the range of localization of particular states in configuration space and on the energy scale, to change the decay rates of given quasistationary states (resonances), and to ensure complete transparency in the usual spaces (the cases of single- and multichannel Schrödinger equations) and on lattices. The surprising role played by special potential wells acting as "carriers" of the selected states is described. They can also be used to carry quasi-stable states through potential barriers, thereby allowing the resonance widths to be controlled. Such a control mechanism is *universal* in nature. All these illustrations have been obtained by using exactly solvable models of the inverse problem. These models form a *complete set*, which makes the quantum mechanics unified (the asymmetry between the direct and inverse problems is eliminated). This review greatly enriches quantum intuition and raises the understanding of wave mechanics to a higher level.

Richard Feynman on quantum mechanics:
"I would like most of you to be able to appreciate
the beauty of our beautiful world, which is what now
makes up most of the true culture of our times.
(There are probably teachers of other disciplines
who would raise some objection,
but I am certain that they are absolutely wrong.)
And perhaps you can not only give this culture its due;
it is not impossible that you yourselves will be drawn
into this great venture on which the human mind has embarked."

1. INTRODUCTION

New exactly solvable models make it possible to formulate a unified quantum theory. Someone once aptly referred to them as milestones of knowledge. Thanks to them, the broken symmetry between the two main components of nonrelativistic quantum theory, the direct and inverse problems, has been to a large degree restored.

1. In the direct problem ($V \rightarrow S$) the interaction potential V determines the various properties of a quantum system S, like the scattering data, the spectral parameters, and so on.

2. In the inverse problem ($S \rightarrow V$) the potentials needed for the system S to have specified properties are determined. The inverse problem is important because it qualitatively reveals new information about the microworld.

Physicists acquainted mainly with the direct quantum problem usually know no more than a dozen solvable models: the square well, the harmonic oscillator, the Coulomb potential, and so on. Many classes of *complete (!) sets* of exactly solvable models can be found in the inverse problem. For example, in the one-dimensional case or after separation of variables such problems can in principle be used to approximate with arbitrary accuracy practically any quantum system. In this review we include in our discussion of some of them clear illustrations which *complement* the well known *Picture Book of Quantum Mechanics* by Brandt and Dahmen for the traditional theory.[13] The exactly solvable models which we now discuss form a bridge between the formerly disconnected halves of the nonrelativistic theory.

Great changes have occurred since our last review of this subject was written.[1] This review permits a look at the still unfamiliar territory of the half of quantum mechanics (the inverse problem) invisible in the direct approach. This was not done earlier, since the quantum inverse problem has been studied mostly by mathematicians and not by physicists. The latter, who often pursue more narrow utilitarian goals (the reconstruction of potentials for specific systems), often do not know about the broad possibilities offered by the mathematical apparatus which has been developed. Mathematicians are usually interested in existence theorems and similar topics. They deal with analytic expressions without graphical illustrations, whereas the latter are very useful for physical applications to clarify the physical meaning of equations and to raise the level of understanding of the phenomena in question to an intuitive level. The relation between the methods of the inverse problem and the supersymmetry approach in nonrelativistic quantum theory has now become clearer (Refs. 3, 4, 15, 19, 21, 29, 35, 36, 41, 43).

We shall discuss this in simple and clear terms, using surprisingly simple qualitative comments to explain our illustrations. The topics that we will cover are:

How to eliminate from the discrete spectrum any given level without disturbing the others or how to create a new level in a given place;

How to shift the range of localization of certain states in space and on the energy scale, using auxiliary potential wells which act as "carriers" of the selected states;

How carriers can be used to change the decay rates of given quasistationary states (resonances) by carrying quasi-bound states through potential barriers, thereby controlling the resonance widths;

How in a similar way the transitions between discrete states can be controlled by varying the overlap integrals;

How to control the transmission in ordinary spaces and on lattices.

A mechanism of making one-channel quantum systems transparent has been found, and the features of the multi-channel potential matrices $V_{ij}(x)$ for which there are no reflected waves at any energy have also been understood. It is surprising that the potential barriers arising in $V_{ij}(x)$ do not spoil the transmission! Paradoxically, they are even needed to obtain 100-percent transmission. And this is true not for particular values of the energy, as in the case of resonance tunneling, but throughout the entire continuum (!).

It has thus become possible to find a simple explanation of much of what has until recently been concealed in the black box of quantum puzzles.

Although all this is demonstrated for exactly solvable models, it is qualitatively true in the general case. It is thus easy for the reader to predict many results without formulas or the use of a personal computer.

We shall extend the didactic comparison of quantum mechanics for spaces with continuous and discrete variables [exotic topics such as solutions of the Schrödinger difference equation and wave motion along lattices of channels and mixed configurations, generalized Schrödinger difference equations of higher (4th, 6th,...) order]. Although by now everyone knows about the amazing solitons, which correspond to *reflectionless potentials*, a peculiarity of wave propagation in a discrete variable is manifested in the impossibility of constructing a transparent local potential $v(n)$. Any local potential well leads to a lowering of the upper boundary of the allowed band (a barrier hanging down from above), which necessarily leads to reflection. This difficulty has been eliminated by introducing minimal nonlocality [potentials $u(n)$ on the next-to-principal diagonals of the matrix finite-difference Hamiltonian], which get rid of these troublesome barriers. The supersymmetry approach has proved useful in this case.[11,36]

It is difficult to exaggerate the importance of quantum control for modern microelectronics, laser technology, quantum optics, and so on. And even more important than the promise of practical applications is the possibility of reaching a new level of understanding of the rules governing the quantum world. The intensive development of quantum theory (a "permanent revolution") involves not only the accumulation of masses of new data, but the bridging of the gap between the latest achievements and the knowledge known to a wide range of specialists and taught in university. In the final analysis the goal of every science is to concentrate the knowledge of nature to a quintessence which occupies as little memory as possible and has the maximum possible force in application. Intuition makes it easier to navigate in the sea of still-unsolved problems in nuclear, atomic, and molecular physics. Even a single look at the pictures presented here would be useful for anyone having anything to do with quantum physics.

2. THE QUALITATIVE THEORY OF CONTROLLING THE RANGE OF LOCALIZATION OF INDIVIDUAL QUANTUM STATES

"... the transition to a single (!) level of the chlorophyll molecule triggers the mechanism for all the life activity on the earth" [P. Raven et al.]

As a clear illustration of the new situation which has arisen in quantum mechanics, in this section we demonstrate how potentials are deformed as individual levels are removed from their spectra, and also the potential perturbations needed to shift given quantum states in space. We have obtained pictures for which we were able to find simple explanations, and it now seems to us that they make up an integral element of the quantum grammar.

Abraham and Moses[17] were the first to give expressions for perturbations which eliminate levels. Strictly speaking, one trivial illustration of the elimination of the ground state from the oscillator has already been given (in the study by Sukumar[15]). Owing to the uniform spacing of the spectrum, if the symmetry is not broken, in this case the curve of the oscillator potential is shifted upward on the energy scale by one interlevel spacing without any distortion of its shape. However, although we did ask ourselves the question, we found it impossible to guess what changes in the same infinite well were needed to make the second or higher levels disappear without using computer graphics.

An attempt was made to use the experience on potential deformation when an individual level is shifted, where it has already been understood (Refs. 1,2,6) what form of perturbation ΔV ensures the shift of selected energy eigenvalues. However, the "annihilation" of a level is equivalent to an infinite (!) number of such shifts. In fact, after moving the "undesired" level to the position of the next level above it, then moving that level to the position of the next level above it, and so on *ad infinitum*, we can check that the state in question has effectively disappeared. However, it is difficult to see how to sum such an infinite number of perturbations.

It proved simplest to explain the essence of the potential transformation in the following way: One node disappears in each of the infinite number of states above the state removed. For this the eigenfunctions must be "shortened by a half wavelength", which is achieved by narrowing the potential well, for example, by moving the left-hand wall toward the right, as in Fig. 2. A shift of the right-hand wall also works. There apparently exists the corresponding symmetric perturbation whose shape is predicted by the "symmetrization" of the curves in Fig. 2.

The first results of the calculations (see Fig. 2) actually appeared incorrect, owing to the violation of the symmetry of the potential when the level is removed. After all, the original well is symmetric and the modulus of the "annihilated state" is symmetric. The shape of the symmetric potential is completely specified by its spectrum. Where does the asymmetry come from? The point here is that the operation of absorbing a level is not unique: the normalization constants, which along with the level positions are the fundamental spectral data, remain as free parameters. In the Marchenko approach which we used the normalizations fix the asymptotic behavior of the eigenfunctions on one side. For simplicity we left these unchanged for all the levels retained, while the normalization of the eliminated state was taken to zero or infinity, thereby introducing the observed asymmetry.

2.1. Elements of the theory of the inverse problem

To remind the reader of the essence of the methods of the inverse problem,[7] let us make a few elementary remarks (so that the introduction to the formalism is as accessible and brief as possible).

Let us consider a potential well with a ladder of discrete bound-state energy levels E_ν with orthonormal wave functions $\psi_\nu(x)$, which we treat as the basis vectors in an infinite-dimensional space. In addition to the energy levels there are also the fundamental spectral parameters, the normalization constants. For the problem on the semiaxis $[0,\infty)$ these can be the coefficients c_ν of proportionality between the functions $\psi_\nu(x)$ normalized to unity and the so-called regular solutions $\varphi_\nu(x)$ having derivative equal to 1 at the point $x=0$:

$$\psi_\nu(x)=c_\nu\varphi_\nu(x). \tag{1}$$

The double set of spectral parameters $\{E_\nu,c_\nu\}$ completely specifies the shape of the infinite potential well. The same functions for fixed value of the coordinate x and any value of E_ν can be treated as other vectors which are also orthonormal, but in this case the property of orthonormality (in the energy variable with weight c_ν^2) is better known as the completeness relation:

$$\sum_\nu c_\nu^2\varphi_\nu(x)\varphi_\nu(y)=\delta(x-y). \tag{2}$$

This relation is satisfied by the solutions for both the original potential and the perturbed potential, except that they are taken at values of the energy of different spectra and orthonormalized with different weights $\mathring{c}_\nu^2$ and c_ν^2 The inverse problem of transforming from the original potential to the perturbed one $\mathring{V}\to V$ reduces to renormalization of the axes: when the weight function is changed, the original eigenfunctions will no longer be orthogonal. Let us recall the standard Gram–Schmidt orthogonalization procedure for the case of two initially nonorthogonal (in the sense of the new weight function) vectors $\mathring{\varphi}_1$ and $\mathring{\varphi}_2$ (see Fig. 1).

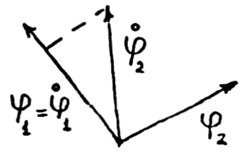

FIG. 1. To make given vectors . $\mathring{\varphi}_1$, $\mathring{\varphi}_2$ orthogonal, the first can be left unchanged, and the projection of the second on the first (in the sense of the scalar product with a new weight) subtracted from the second.

As the first axis φ_1 of the new system we take the first unperturbed axis $\mathring{\varphi}_1$, and we construct the second axis from the second unperturbed axis, only subtracting from it the excess (the part parallel to $\mathring{\varphi}_2$) to make it orthonormal with the new weight.

$$\varphi_1=\mathring{\varphi}_1;\quad \varphi_2=\mathring{\varphi}_2-K_{21}\mathring{\varphi}_1.$$

The orthogonalization can be begun with any vector. The continuous analog of this procedure, when the values of x play the role of the indices numbering the axes, and orthonormality is required in the energy variable (here the completeness relation plays the role of the orthogonality conditions), is

$$\varphi_\nu(x)=\mathring{\varphi}_\nu(x)+\int K(x,y)\mathring{\varphi}_\nu(y)dy, \tag{3}$$

where the limits of integration are 0 and x in the Gel'fand–Levitan (GL) approach, and x and ∞ in the Marchenko (M) approach. The coefficients K connecting the functions of the original potential to those of the new potential are determined by the condition that $\varphi_\nu(x)$ be orthonormal to all vectors with "numbers" ($y<x$ or $y>x$ in the GL and M approaches, respectively) already used in constructing the new axes as long as they went to the point x (to the left or right in the GL and M approaches). These equations for K are simultaneously the equations for the inverse problem [with limits of integration as in Eq. (3)]:

$$K(x,y)+Q(x,y)+\int K(x,t)Q(t,y)dt=0, \tag{4}$$

and the kernel Q is constructed from unperturbed functions with the old and new spectral parameters:

$$Q(x,y)=\sum_\nu c_\nu^2\mathring{\varphi}_\nu(x)\mathring{\varphi}_\nu(y)$$

$$-\sum_\mu \mathring{c}_\mu^2\mathring{\varphi}_\mu(x)\mathring{\varphi}_\mu(y)+\int \dots . \tag{5}$$

The first sum here is specified by the perturbed parameters E and c, and the second is specified by the original parameters. In our problems there will not be any changes in the continuum, so that the integral corresponding to changes in the scattering data vanishes.

If the spectral parameters of a finite number of bound states are changed, the corresponding terms in the left- and right-hand sums cancel for the others, so that a finite number of terms factorized in the "numbers" x and y remains in Q.

And the equation of the inverse problem with a degenerate kernel of this type reduces to a system of a finite number of algebraic equations. The K found from them is used to determine the new (perturbed) potential.

The Gram–Schmidt orthogonalization procedure can be started from any original vector. Since the "number" of the vector serves as the coordinate in the inverse problem, it is possible to begin the orthogonalization from the asymptote $x=\infty$ (the Marchenko approach) or from the origin (the Gel'fand–Levitan approach).

2.2. The transformed potentials

Let us consider the kernel Q of the integral equation of the inverse problem (in the Marchenko approach; see, for example, Ref. 7):

$$K(x,y)+Q(x,y)+\int_x^\infty K(x,t)Q(t,y)dt=0. \tag{6}$$

It is essentially the Schrödinger equation "turned inside out." Instead of the procedure followed in the direct problem, where some potential $V(x)$ is inserted into the ordinary Schrödinger equation, which is then solved for the physical properties of the given quantum system, in the inverse problem we can insert the *desired* spectral parameters into the kernel Q:

$$Q(x,y)=\sum_{\nu}^{N}\mathring{\psi}(k_\nu,x)\mathring{\psi}(k_\nu,y)-\sum_{\mu}^{M}\mathring{\psi}(k_\mu,x)\mathring{\psi}(k_\mu,y), \tag{7}$$

where the constants $k_\nu^2=E_\nu$ and $k_\mu^2=E_\mu$ determine the energy levels in the new potential V and in the original potential $\mathring{V}$ (of which there are an infinite number $N=M=\infty$ in our case), and the integral [shown only by the integral sign in (5)] pertains to continuum states, of which there are none in infinite wells. The solutions $\mathring{\psi}_\nu$ in the first sum correspond to the Schrödinger equation with the original potential, but are taken at the energy eigenvalues for the perturbed well and with the corresponding new normalization factors. If the functions on the left-hand side for the levels retained are chosen to be the same as those on the right-hand side, this will correspond to choosing the former normalization constants for them. Then when a finite number of wells are removed from the original well $\mathring{V}$, in (7) there will remain only the terms of the second sum corresponding to them, since the parameters of the other levels are assumed to be unchanged and the corresponding terms in the first and second sums cancel each other. The equation of the inverse problem with this *degenerate* kernel Q reduces to an algebraic equation. It is easy to solve, i.e., to find K, which in turn is the kernel of the integral operator of a generalized shift transforming the wave functions of the original unperturbed system $\mathring{\psi}$ into the functions of the system with the new potential $V(x)=\mathring{V}(x)-2K'(x,x)$. When one level is annihilated, we have

$$V(x)=\mathring{V}(x)-2\frac{d}{dx}\left[\frac{\mathring{\psi}_\mu^2(x)}{1-\int_x^\infty \mathring{\psi}_\mu^2(y)dy}\right]. \tag{8}$$

In Fig. 2 for the cases of original potentials which are an oscillator on the entire axis and a linear well on the semiaxis we show the perturbed potentials when the first, second, and third levels are removed.

The shape of the bottom of the potential well is determined by the fact that after it is narrowed a finite number of levels *below* the one removed are raised, and these must be shifted back down according to the rule discussed in Refs. 2 and 6. *The number of local minima is equal to the number of maxima of the modulus of the wave function of the next state below the "spectrum hole,"* since it is most sensitive to perturbations where the probability of finding the particle at the given level is largest. The sinking of the lowest states is not manifested so clearly in the shape of the perturbation (it only "modulates the carrier frequency" of the most energetic of the group of states under the spectrum hole which is created).

When *two* levels are removed, the narrowing is about twice as great.

Now we can very reliably predict that *to generate a new level it is necessary to widen the well* and keep the finite number of states located below the new level in their original locations by pushing down near the maxima of the moduli of their wave functions.

The perturbation of the potential tends to zero at large $|x|$, as can be verified by using the l'Hôpital rule to resolve the indeterminate quantity 0/0.

When one of the complete set of spectral parameters $\{E_\lambda,c_\lambda\}$ or $\{E_\lambda,M_\lambda\}$ is varied, in the Marchenko formalism all the eigenfunctions change (including those of the states whose spectral parameters were not changed). See the discussion below regarding the choice of normalization M_λ instead of c_λ.

The results for potentials on the semiaxis are relevant also to symmetric three-dimensional problems. Of course, it should be noted that in this case we are dealing only with levels corresponding to a certain value of orbital angular momentum l: for example, in the annihilation of an s level the levels with other $l>0$ will be shifted. To keep them fixed it is necessary to introduce into the potential some dependence on additional variables, for example, a nonlocality in the angles. The results on the entire axis are also applicable to the three-dimensional case, but for potentials independent of two of the three variables.

The simple explanation of the results described above allows one to make rather reliable predictions about the qualitative characteristics of the potentials which must be obtained in other formulations of the problem.

If in the Marchenko approach in the equations of the inverse problem the limits of integration over the spatial coordinate $[x,\infty)$ are replaced by $(-\infty,x]$, the resulting potentials and wave functions are the mirror images relative to the point $x=0$.

In the Gel'fand–Levitan approach for the oscillator potential on the semiaxis (with limits of integration $[0,x]$) the vertical potential wall at the origin must remain unchanged,

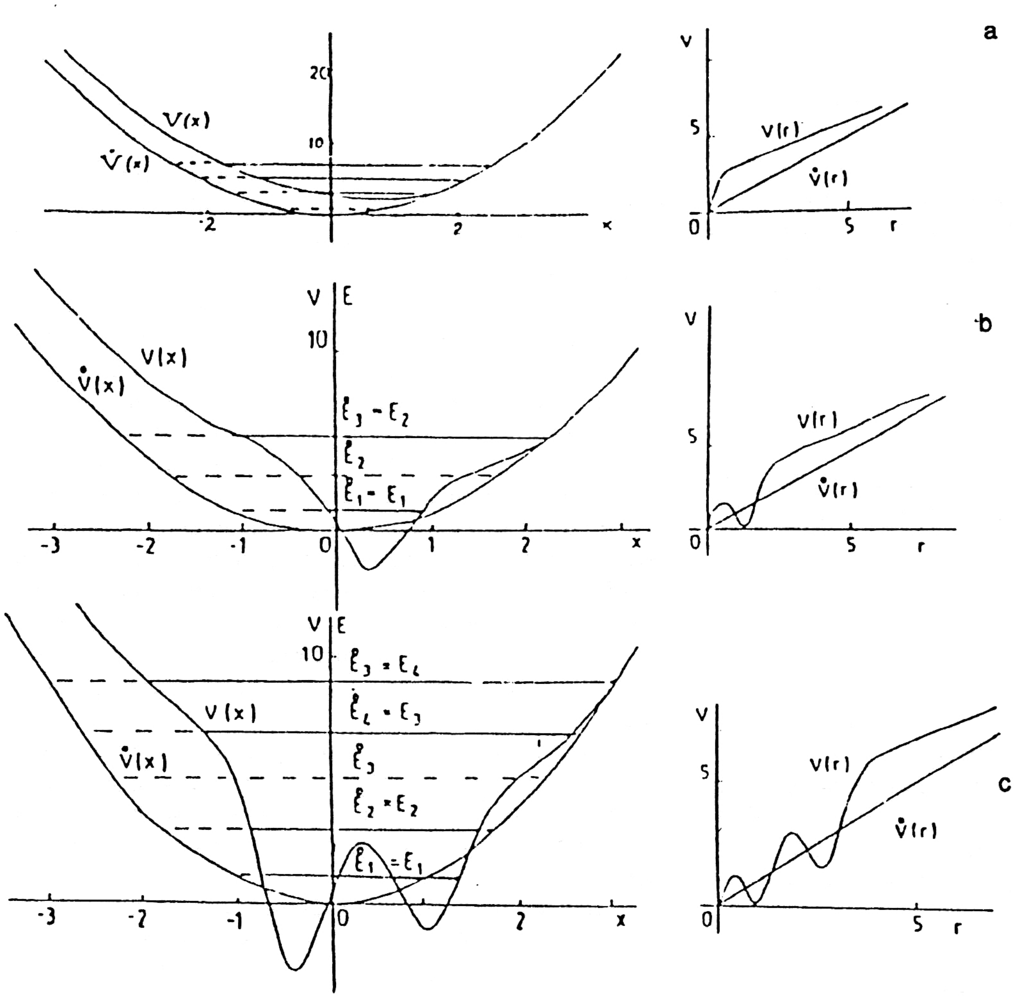

FIG. 2. (a) Change of shape of the oscillator and the linear well when the first level vanishes. The well becomes narrower, and the other states are shortened by half an oscillation. (b) Change of shape of the oscillator and the linear well when the second level vanishes, as in Fig. 2a, but with an additional small well which returns the first level, raised when the well is narrowed, to its original location. (c) Change of shape of the oscillator and the linear well when the third level vanishes. The additional small wells return the first and second levels to their original locations. The narrowing of the well is caused by the decrease of the number of oscillations of the eigenfunctions of states lying above the eliminated level; cf. Figs. 3–5, where well deformation is induced by the creation of new levels.

and the right-hand edge of the infinite well will move. This has been confirmed by calculations using the expression (see Fig. 2 for the case of the linear well)

$$V(x)=\mathring{V}(x)+2\frac{d}{dx}\left[\frac{\mathring{\psi}_\mu^2(x)}{1-\int_0^x \mathring{\psi}_\mu^2(y)dy}\right]. \tag{9}$$

If the perturbed well is continued symmetrically to the left-hand semiaxis, in the resulting wells only the levels with odd states will coincide with the levels of the symmetric oscillator, since only they are simultaneously the levels of the well on the semiaxis. Therefore, the shape of the right-hand wall of the perturbed potential well on the semiaxis must also differ from the case where "the same" level is eliminated in the Marchenko approach on the entire axis, since only the odd states of the symmetric well enter into the game on the semiaxis.

If, instead of annihilating existing levels, *new states are generated*, it is natural to expect a corresponding *widening* of the potential wells. The case of the generation of a level in the finite square well is shown in Fig. 9, and the case of a linear well $v(r)=r$ is shown in Figs. 3–5. Here in the equations for the potential of the type (8) in the denominator it is necessary to replace the minus sign by a plus sign. In this case it is more complicated to use expressions of the type (8), since the *unphysical* solutions in the original well at values of the energy *between* the bound states are needed (this is not complicated for the square well).

The ground state can be eliminated symmetrically, using

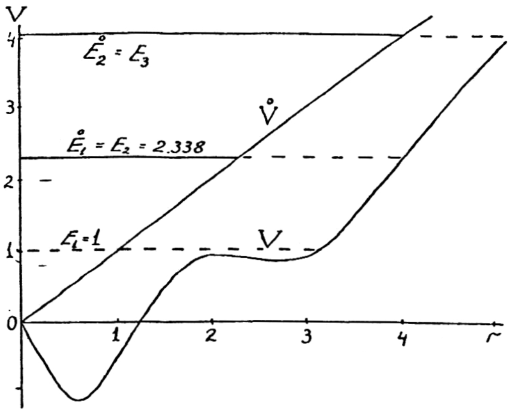

FIG. 3. Addition of a new level $E_1=1$ (below the ground state) to the spectrum of the linear well. The well broadens, and the shape of its bottom ensures that the created level has the correct location.

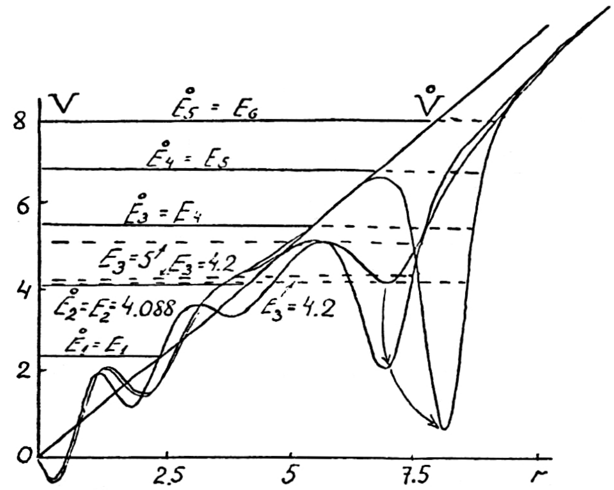

FIG. 5. Addition of a new level $E_3=5$, or $E_3=4.2$, or $E_3=4.1$ to the spectrum of the linear well. The change of the well shape as the level E_3 approaches $E_2=3$ should be noted.

the supersymmetry technique.[2] It would also be interesting to generalize this work to multichannel systems.

2.3. Elimination of a level E_μ as the limit $c_\mu \to 0$ ($M_\mu \to 0$)

Changes of the value of the normalization constant c_μ (M_μ) can be related to the elimination of the corresponding energy level from the spectrum. From the equation for Q,

$$Q(x,y)=c_\mu^2\,\mathring{\varphi}_\mu(x)\mathring{\varphi}_\mu(y)-\mathring{c}_\mu^2\mathring{\varphi}_\mu(x)\mathring{\varphi}_\mu(y), \tag{10}$$

it follows that when the normalization c_μ is chosen to be zero the expression for the kernel Q coincides with that obtained when the μth level is eliminated:

$$Q(x,y)=-\mathring{c}_\mu^2\mathring{\varphi}_\mu(x)\mathring{\varphi}_\mu(y). \tag{11}$$

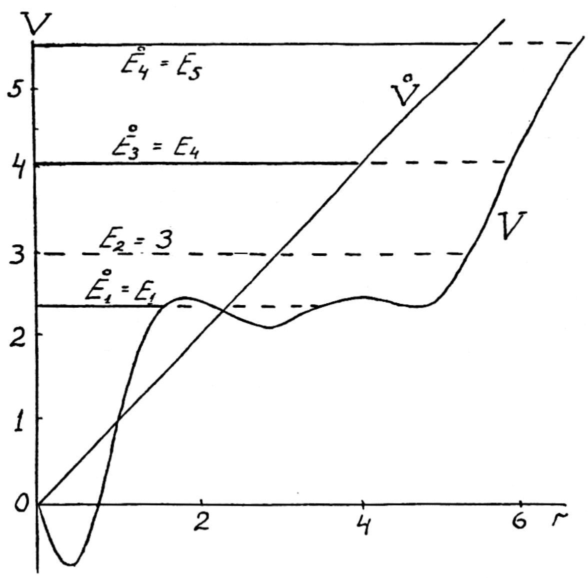

FIG. 4. Addition of a new level $E_2=3$ to the spectrum of the linear well. The shape of the well bottom ensures that the pair of low-lying levels has the correct location.

It would seem to be impossible to imagine how the *continuous* process of changing the normalization of *stationary levels* can be used to achieve the *discontinuous* result of *elimination* of a level. A typical picture for a choice of normalization close to zero is shown in Fig. 6. In this case, the characteristic "icicle trap" in the excited potential is obtained in the Marchenko approach at very small M_2.

The shape of the icicle is such that favorable conditions are created in it for a standing half-wave of the selected state, while for all the other states there is cancellation owing to the destructive interference of waves multiply reflected from the walls, which becomes stronger, the farther the icicle is moved to the side. This property of "slidability" of a selected bound state from the tower of states when the corresponding normalization is changed is a typical reaction of a quantum system (in the case of a one-level system the entire well moves as a whole). At the same time, the barrier narrowing the original oscillator well and decreasing the num-

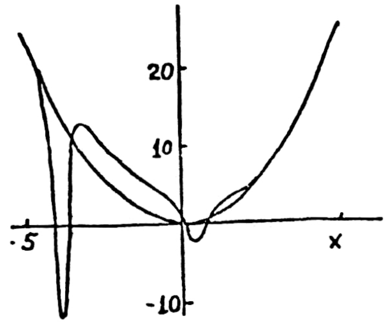

FIG. 6. Normalization constant M_2 close to zero for the second level produces in the perturbed potential a small, narrow well (an "icicle trap") carrying the range of localization of this state to minus infinity as $M_2 \to 0$, which can be viewed as the gradual "elimination" of this state. States lying higher in energy each have one node under the barrier separating the icicle from the main well. Inside the icicle they undergo their final oscillation with amplitude which decreases as the icicle moves away.

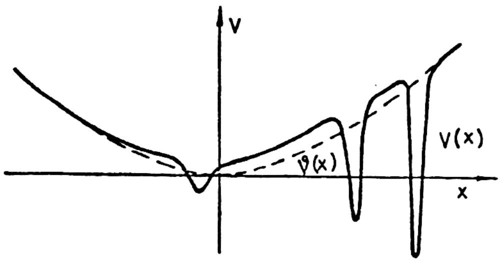

FIG. 7. A combination of two icicles produces the interchange of the localization of two states (the second and third) as the corresponding normalization parameters M_2 and M_3 are increased.

ber of nodes of the states above the selected one in the main part of the well grows. The last nodes (one for each of these states) move inside the barrier, dividing the icicle and the main well. In the limit these last nodes move to infinity. When several levels are removed (their normalizations taken to zero), several icicles appear, with their relative locations depending on the ratios of the normalization constants. In particular, icicles can be located on one side or on both sides of the main well. For large M_μ (or small c_μ) the icicles are formed on the right.

This sheds additional light on the perturbation of potentials by varying the normalization constants, which we seemed to have already fully understood (see Refs. 2 and 6). For example, a change of the reduced widths in an infinite square well gives a perturbed potential with "embryonic" icicles which simultaneously play the role of correcting the positions of the lower levels on the energy scale. In wells whose walls are not vertical these (one or several; see Fig. 7) embryos can be shifted by any amount. Here each carries one node of the corresponding wave function.

The rightward shift of the localization of the states corresponding to the first and second levels of an originally finite square well with five bound states is shown in Figs. 8a and 8b. Here the small carrier well has a soliton-like shape (thus the continuum of the original potential is unchanged because the soliton potential is reflectionless). The characteristic "potential tooth" appeared to compensate for the

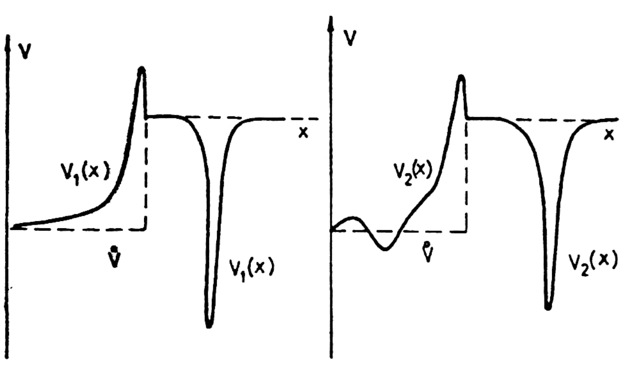

FIG. 8. (a) Shift to the right of the ground state from a square well having five bound states. (b) Shift of the localization of the second state. (in Fig. 6.) See the transformation of the bottoms of the square well and the oscillator.

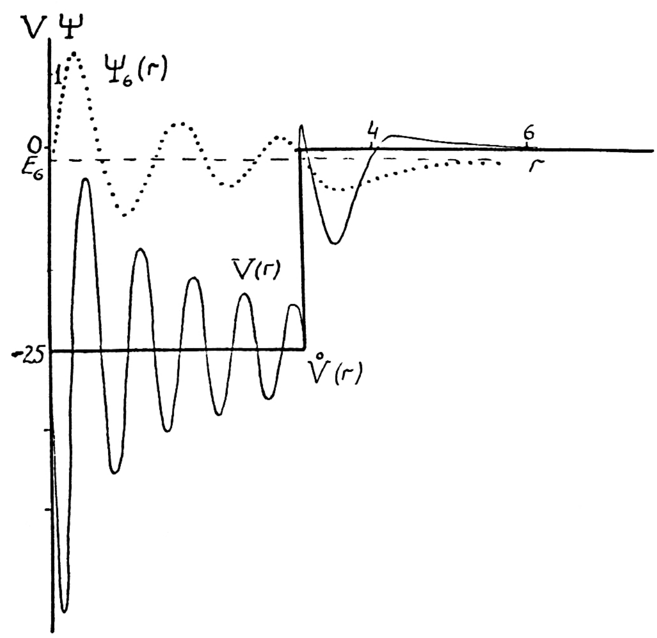

FIG. 9. Creation of a new sixth level, shown by the dashed line, in a finite square well of depth 25 and width π having five bound states. Cf. the transformation of the linear potential in Figs. 3–5. The dotted line shows the normalized wave function of the created state.

smoothing of the sharp edge of the original square well, so that the former scattering properties remain unchanged in the continuum. This should be compared with the corresponding perturbations of the bottom of the oscillator well (Figs. 2a and 2b).

In Fig. 9 we show an example of the generation of a new (the 6th) level in a finite square well. The oscillatory perturbation of the flat bottom of the potential is needed, on the one hand, to lift the old levels to their former locations; otherwise they would have been lowered, owing to the widening of the original well due to the additional small well on the right. On the other hand, the growth of the potential oscillations on the left serves to pile up the created state near the origin, since we have chosen the derivative $\Psi_6'(0)$ to be large. The barrier (the "potential tooth" at the discontinuity of the original square well) between the main well and the auxiliary well is needed to keep the spectral function of the continuum unchanged. This should be compared with Figs. 3–5, which show the deformations of an infinitely deep linear well when new levels are inserted into the spectrum.

The equations for changing the normalization constants were also obtained by the supersymmetry technique, long after they were obtained by the method of the inverse problem (see Ref. 3, the studies from 1988 and 1991).

It is interesting to extend these results to the multichannel case and, possibly, to problems with difference equations (with a discrete variable), in particular, to "higher-order Schrödinger equations" (see Ref. 1).

2.4. The pulling of selected states through a barrier

Intuition makes it possible to predict qualitatively the properties of quantum systems without the use of formulas

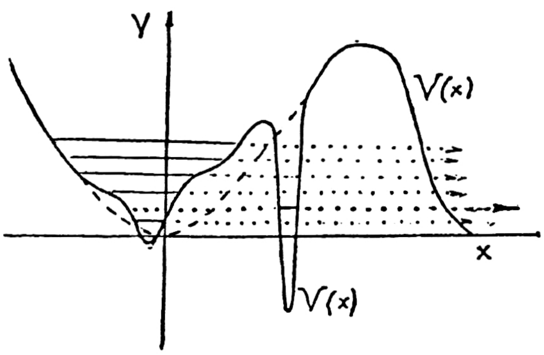

FIG. 10. Distortion of the shape of the initial (dashed line) oscillator well when the normalization constant M_2 of the second state is increased. As in Fig. 6, the additional narrow well carries the state in question the farther to the right, the larger the selected value of M_2. If in addition one of the infinite walls of the initial oscillator is bent in such a way that it is transformed into a potential barrier of finite size, the bound states are transformed into quasistable states, and the transport of the selected state to the edge of the barrier increases the probability for it to decay by a given factor.

and computers. Let us study the picture (Fig. 10) of the transformation of an oscillator into a well with a finite barrier in place of the infinite wall on the right as the normalization constant of the second level—the coefficient M_λ of the decreasing exponential in the asymptotic behavior of the wave function on the right-hand side—increases. The auxiliary narrow well carries the selected state the farther, the stronger the change of the selected value M_2.

If one of the infinite walls of the oscillator well is transformed into a finite potential barrier, the bound states are transformed into quasistable ones (resonances). Although this drastically changes the behavior of the system, the special role of the auxiliary small well as a carrier of selected states is not lost in the corresponding fine tuning of its shape. Quantum intuition helps here (we do not have to solve the eigenvalue problem for the quasistationary states, which would require a search for resonances in the complex k plane). It is clear that the transport of the state under the barrier closer to the outer edge makes decay easier and increases the width of the selected resonance without changing the others. It is natural to assume that here the lower-lying states will not be strongly perturbed. The auxiliary narrow well shifting the range of localization of the selected state suggests that this phenomenon can be used to control the matrix elements of transitions between different bound states.

2.5. Controlling transitions

"I am always being accused of discovering the obvious. One accuser ... even went so far as to refer to my work as a blinding flash of the obvious. ...I am satisfied with this characterization." [Thomas J. Peters]

The understanding of the phenomenon of wave-function localization in a restricted range of a set of potential wells can be used to generate desired transitions between levels. For example, in the extreme case it is possible to arbitrarily specify a lattice of levels and require that a transition occur between the ith and jth states: only the matrix element of some local operator O is nonzero ($\langle j|O|i\rangle$). Such a quantum system is easily constructed by placing the selected levels in one partial small well, and each of the others in separate wells separated by barriers of negligible transmission. The absence of overlap of the functions eliminates undesired transitions. A similar procedure can also be realized in the more general case when definite ratios of the transition probabilities for an arbitrary discrete spectrum are specified. The "partial reduced widths" of the states in the individual wells provide additional "levers" for controlling the values of the matrix elements, as these widths can be regulated according to the rules for changing the normalization spectral parameters by choosing the shape of these partial small wells.

3. CHANGE OF THE REDUCED WIDTHS IN THE MULTICHANNEL APPROACH

The question of what potential perturbations in the one-channel case lead to a change of the reduced width (the derivative at the edge of an infinite square well) of an arbitrarily chosen state was discussed in Refs. 2 and 6. Here we shall give an example of controlling these fundamental spectral parameters in a multichannel system. The figures given below complete the set of "quantum pictures."

We shall decrease the modulus of the derivative in the ground state, $\psi_1'(0)$, of the wave function of the second channel at the left-hand infinite potential wall of the original two-channel system with constant interaction matrix:

$$\mathring{V}_{11}(x)=\mathring{V}_{22}(x)\equiv 0; \quad \mathring{V}_{12}(x)=\mathring{V}_{21}(x)\equiv 0.3.$$

In Figs. 11–13 we show the resulting changes of the

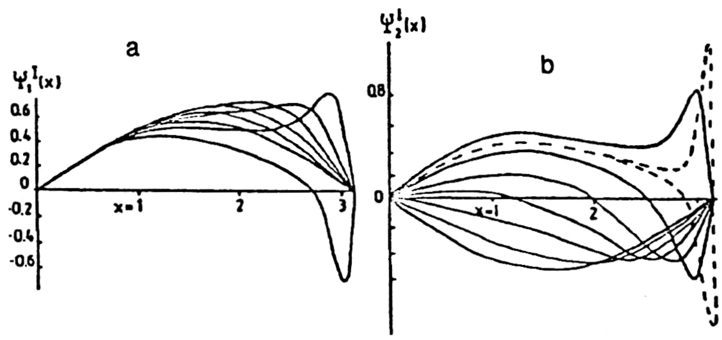

FIG. 11. Change of the wave functions of the ground (a,b) state of the two-channel problem when the derivative of the ground-state function in the second channel on the left-hand boundary, $\psi_2'(0,\lambda_1)$, increases from its initial negative value $\mathring{\gamma}_2=-1/\sqrt{\pi}$ according to the law $\mathring{\gamma}_2+0.2m$; $m=1,2,...,6$ (solid lines). The dashed line shows the intermediate function, so that the behavior of the function as the node disappears can be seen more clearly. The waves "overflow" from one channel to the other.

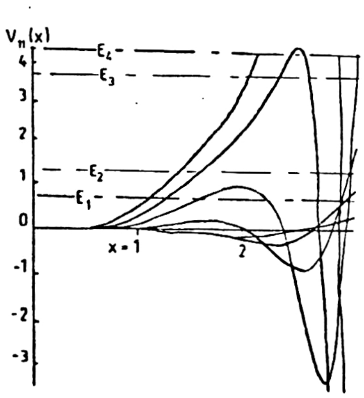

FIG. 12. Perturbing additions to the interaction matrix element V_{11} that correspond to the increase of the derivative of the ground-state function in the second channel, as shown in Fig. 11b. The horizontal broken lines show the positions of the energy levels—the two lowest doublets.

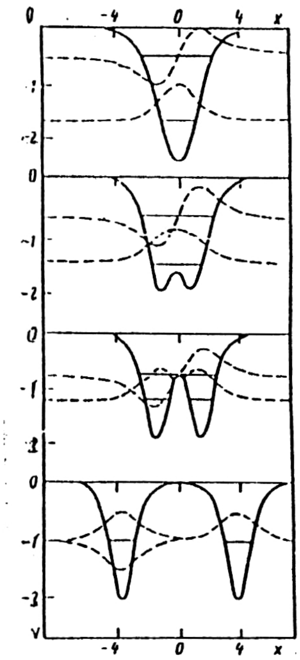

FIG. 14. Examples of reflectionless potentials with two bound states [Refs. 34 (1980) and 39 (1986)]. The one-level well is similar to the upper well with two levels, but is not as deep.

channel functions and of the interaction matrix.

At first (for a small increase of the derivative at $x=0$) the behavior of the function in the second channel is reminiscent of the one-channel case: the function seems to be "extruded" from left to right, except now without change of the normalization: part of it "overflows" into the first channel, owing to the coupling $V_{21}(x)$. Here the changes in the function of the first channel should not affect its left-hand side, since the derivative at the point $x=0$ remains fixed. This is manifested in the growth of $\psi_1(x)$ in the center of the well. As the derivative at the point $x=0$ in the second channel becomes positive and grows further on the left there occurs a reverse "suction" of the wave from the first channel in the central part of this segment. This now causes $\psi_1(x)$ to decrease at the center of the well.

At the same time the negative part of the wave function of the second channel is squeezed and, owing to the channel coupling, it goes into the first channel, where a swelling of the wave function is formed on the right. In contrast to the one-channel case, here it is possible for the wave function and the derivative of the second channel wave function to vanish simultaneously on the left edge for $x=0$; see Fig. 11. Owing to the coupling to the first channel, where the derivative is nonzero at $x=0$, $\psi_2(x)$ acquires nonzero values in going away from the point $x=0$.*

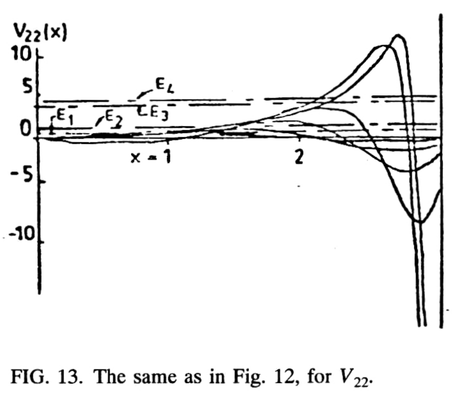

FIG. 13. The same as in Fig. 12, for V_{22}.

We have become acquainted with some features of eigenfunctions with changed reduced width, but our goal is *to understand how the form of the elements of the interaction matrix $V_{\alpha\beta}(x)$ controls the spectral parameters.* For small changes of γ_2 the potential $V_{22}(x)$ in the second channel is reminiscent of the corresponding one-channel case (see Fig. 13, the barrier on the left and the small well on the right). When γ_2 is increased further, this barrier and well move to the right and squeeze the negative part of the function $\psi_2(x)$. In the first channel $V_{11}(x)$ on the left is close to zero (Fig. 12), which corresponds to the fact that the slope of the function $\psi_1(x)$ does not change on the left. The barrier and the well in $V_{11}(x)$ on the right make it possible for a swelling of the function $\psi_1(x)$ to be produced on the right.

4. REFLECTIONLESS ONE- AND MULTI-CHANNEL SYSTEMS

Unexpected features

It recently[25] became possible to understand the features of multichannel potential matrices $V_{ij}(x)$ which do not produce reflected waves at any energy. Surprisingly, the potential barriers appearing in $V_{ij}(x)$ do not spoil the transmission! Paradoxically as it may seem, they are even necessary for 100-percent transmission. And this is true not at particular energies, as in resonance tunneling, but over the entire continuum (!).

Nearly everyone now knows about the remarkable properties of solitons, which are associated with *reflectionless potentials* (see the examples in Figs. 14–16).

This has already been stored in the bank of valuable elements of quantum intuition (although complete understanding is still far off). However, one-channel systems are

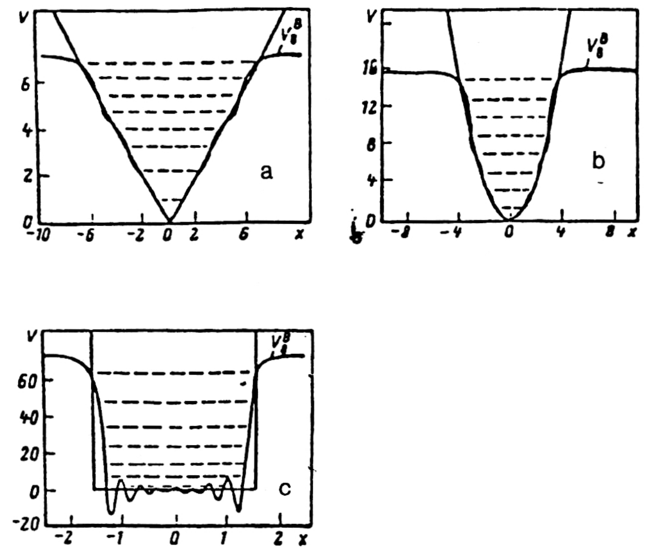

FIG. 15. Transformation of (a) the linear well, (b) the oscillator, and (c) the square well in such a way that they become transparent.

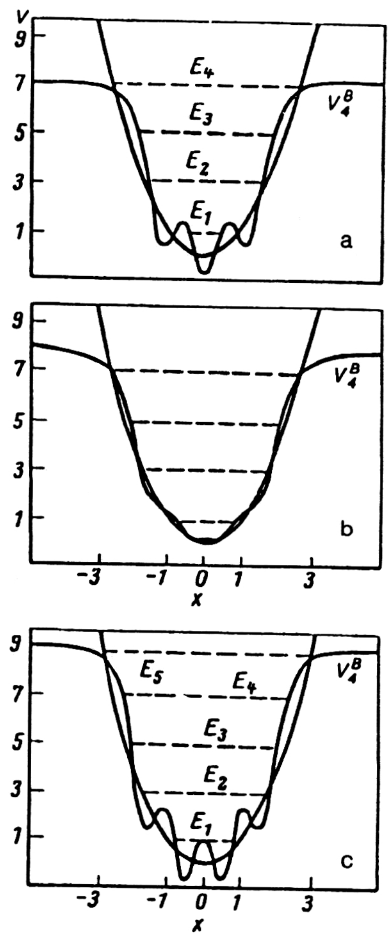

FIG. 16. The quality of the approximation of the lower part of the infinite oscillator using transparent wells of finite depth depends strongly on how close the upper level of the transparent well is to the continuum. The oscillator level closest to the edge of the finite well is particularly sensitive to the "bending back" of the infinite walls, and large oscillations of the potential are required to leave it in its original place.

only a limitingly simple case of quantum objects. Many discoveries remain to be made in the understanding of the real world of multichannel processes and structures.

To the marvels of quantum systems we can now add multichannel interaction matrices which are absolutely transparent for incident waves in any channel. And the approach of the inverse problem makes it possible to obtain such matrices and the corresponding solutions in closed form. Therefore, here we shall show what form they have, both analytically and graphically ("pictures" of reflectionless matrix potentials). In addition, we have succeeded in finding a simple explanation of the remarkable properties of these matrices related to the features of their seemingly complicated structure. As a guess, we use an analogy between this phenomenon and resonance tunneling, for which one of the present authors (B.Z.) earlier has given a nontraditional explanation,[2] and also the idea of passing around a barrier in equations of higher order.[1]

There is an "eternal" contradiction between "pure" and applied science. The computational details involved in treating a complicated system which actually exists make it difficult to get a clear picture of the phenomenon of interest. And an idealized model, although it gives a crystal-clear demonstration of a new phenomenon, may correspond to a system which is unrealizable in practice. Here we shall actually discuss an idealized model. However, having understood the essential physics in it (the elements of the underlying theory of multichannel transmission), it will be easier to seek the same features in complicated real situations.

4.1. One-channel examples

Let us first recall the one-component (scalar) case.

In Ref. 25 it was explained how to approach the understanding (at an intuitive level) of one-channel scattering. It turned out that for this one can use pictures of the reconstruction of infinite potential wells from the lower part of the discrete spectrum, using transparent wells of finite depth; see Figs. 15 and 16 (Refs. 34 and 39).

For a well with a single level there is a family of transparent potentials with two free parameters. This is the situation of a level E and normalization M, the factor in the decreasing exponential in the asymptote of the bound-state wave function to the right of the well, which fixes the location of the potential well on the x axis ["slidability": the potential "slides" along the x axis as M is changed, since in the asymptotic behavior of the bound-state wave function $\psi(x) \overset{x\to\infty}{\longrightarrow} M\exp(-\kappa x)$ a shift in x by Δx is equivalent to the renormalization $M \to M\exp(-\kappa\Delta x)$]. The class of reflectionless potentials with two levels is characterized by *four* free parameters (E_1, E_2, M_1 and M_2). Here the normalizations determine not only the overall location of the potential on the x axis, but also the relative separation of its two "components." The three-level case is, naturally, described by six parameters. Of course, the potentials in Figs. 14–16 are symmetric about $x=0$, and so their shape is specified only by the level positions (without specifying the factors M; Ref. 2). As the levels get closer the individual soliton wells become

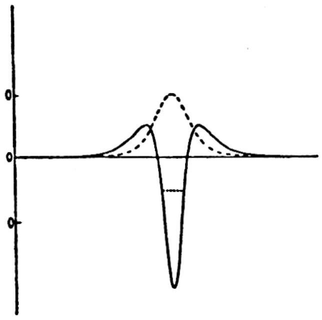

FIG. 17. The initial repulsive potential (dotted line) can be deformed in such a way that a bound state appears while the reflection coefficient $R(E)$ remains unchanged [Ref. 15 (1987)].

more and more similar to each other and move to the sides, ensuring twofold degeneracy in the limit.

It is also instructive to become acquainted with pictures showing *what changes must be introduced, for example, into the square well, the linear well, and the oscillator to make them transparent.*

We see from Figs. 15 and 16 that the more abruptly the initial potential changes or *the closer the level of a well to the continuum of the transparent well, the larger the corrections needed to make the well transparent.* This can be explained as follows. In the "bending back" of infinite potential walls the levels below the continuum which appears must drop down the more strongly, the closer the level to threshold. In order to return the levels to their former positions, it is necessary to use the correction algorithms discussed above (to create bumps which push upward at the maxima of the modulus of the wave function). It can be said that it is as though the additional oscillations "cure" the perturbations of the continuum.

Let us give another example of the analogous construction of a potential which preserves the reflection properties of the original potential barrier (Fig. 17). By varying the normalization M of the bound state it may, of course, be possible to spoil the symmetry of the potential, by shifting the created dip relative to the original barrier.

4.2. A system of strongly coupled Schrödinger equations

Modern quantum theory "is becoming more and more a multichannel theory": the description of any structure or process (multidimensional and multiparticle) can be reduced to systems of coupled Schrödinger equations for partial channels.

Let us consider a multichannel system of coupled one-dimensional Schrödinger equations (Refs. 2, 5, and 6):

$$-\psi_i''(x)+\sum_j V_{ij}(x)\psi_j(x)=E_i\psi_i(x). \tag{12}$$

Here $E_i=E-\varepsilon_i$ are the channel energies, and ε_i are the values of the thresholds of the continuum in the individual channels. The system (12) establishes a direct link between the interaction matrix $V_{ij}(x)$ and the channel wave functions $\psi_i(x)$ determining the observed properties of the corresponding model quantum object. In general, looking at (12), it is difficult in practice to guess these properties.

However, the approach of the inverse quantum problem can be used, where the desired scattering data or spectral parameters are specified *a priori* (Refs. 1, 2, and 7). For example, it can be required that the potential matrix in the system (12) be reflectionless, so that its scattering properties are the same as those in a system without interaction for the uncoupled equations describing free wave motion (complete transmission at any value of the continuum energy), but here there are bound states. For this we use the equations of the inverse problem in the Marchenko approach (see Refs. 2 and 6, for example) with zero initial interaction matrix $\mathring{V}_{ij}=0$:

$$K_{ij}(x,x')+Q_{ij}(x,x')+\int_x^\infty K_{il}(x,x'')Q_{lj}(x'',x')dx''=0 \tag{13}$$

$$Q_{ij}(x,x')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\ (ik_ix)[\mathring{r}_{ij}(k_i)-r_{ij}(k_i)]\exp(ik_jx')dk_i + \sum_\nu^N \exp(-\kappa_i^\nu x)M_i^\nu M_j^\nu \exp(-\kappa_j^\nu x'). \tag{14}$$

Putting into Q the desired spectral parameters of the bound states [the binding energies $E_\nu=\varepsilon_i+(\kappa_i^\nu)^2$ and the normalization constants M_i^ν] and also the scattering data (the reflection coefficients), from the equations of the inverse problem we find the corresponding K, which, in turn, determines the desired interaction matrix:

$$V_{ij}(x)=-2\frac{d}{dx}K_{ij}(x,x). \tag{15}$$

In our case (free motion in the original system) the integral term in the kernel Q vanishes, since we have specially chosen the difference of the scattering functions (the reflection coefficients) $\mathring{r}-r$ to be zero. As a result, we have a degenerate (factorized in the variables) kernel of the integral equation (12), which reduces to algebraic equations,[6] whose solution in the special case where a single bound state is created can be expressed in terms of the Jost solution of free motion in uncoupled channels, $\mathring{F}_{ii}(x)=e^{(-\kappa_i x)}$, at the energies E_b of the created bound state:

$$K_{ij}(x,y)=-\frac{M_i\mathring{F}_{ii}(x)M_j\mathring{F}_{jj}(y)}{1+\sum_m\int_x^\infty[M_m\mathring{F}_{mm}(y)]^2dy} \tag{16}$$

with a sum over channels. The energy E_b, the normalization constants M_1 and M_2, and the difference of the channel thresholds are free parameters. According to (15) and (16), we have a four-parameter family of real reflectionless interaction matrices with a single bound state, one of which is shown in Fig. 18. The corresponding Jost solutions of the transparent system at arbitrary energies have the form

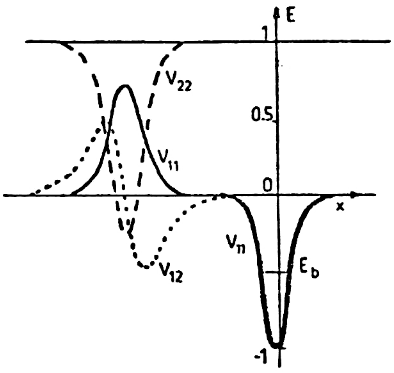

FIG. 18. Transparent interaction matrix with one bound state at $E=-0.5$; $M_1=1, M_2=0.001$. The thresholds at which the channels open are $\varepsilon_1=0$ and $\varepsilon_2=1$. Reflection from the barrier in $V_{11}(x)$, shown by the solid line, is suppressed by waves from the "decay" (from the second to the first channel) of states "confined" by the well $V_{22}(x)+\varepsilon_2$, shown by the dashed line. At low energies the waves which would have to tunnel through the barrier in $V_{11}(x)$ partially go around it via the second channel, owing to the channel coupling $V_{12}(x)=V_{21}(x)$ (dotted line), which vanishes at the top of the barrier.

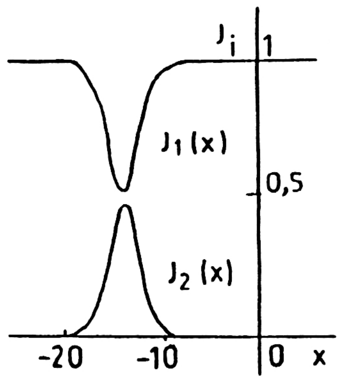

FIG. 20. Partial fluxes J_i in the two channels at an energy for which only the first channel is open (incident waves only in the first channel). Only the total flux $J_1+J_2=$const is conserved. In the vicinity of the soliton-shaped well in $V_{11}(x)$ there is no flux in the second channel.

$$F_{ij}(x)=\mathring{F}_{ij}(x)+\sum_m \int_x^\infty K_{im}(x,y)\mathring{F}_{mj}(y)dy. \qquad (17)$$

Some of the Jost solutions (vector solutions for one open channel and matrix solutions for two) are shown in Fig. 19. The right-hand subscript of the functions labels the channel containing the incident wave, i.e., it fixes the boundary conditions. The left-hand subscript indicates the channel in which the given wave occurs.

The well on the right in V_{11} has the form of a one-level reflectionless well, but the appearance of the barrier in the potential V_{11} was completely unexpected and would certainly seem to spoil the reflectionless nature of the interaction. Actually, *this barrier does induce reflection, but it is essential for canceling out other undesired waves.*

The phenomenon of tunneling lies at the very heart of quantum mechanics. Several Nobel Prizes have already been awarded for advances in the understanding and practical application of this phenomenon. Up to now this has mainly concerned one-channel processes. However, the real world is considerably richer than one-channel phenomena. Although much in this area remains puzzling, attempts are now being made to open up the unlimited vistas of multichannel physics. Here we shall take a step in the understanding of the mechanism by which waves go around barriers, using extra (channel) degrees of freedom (see Chap. 2 of Ref. 7) in order to avoid the difficulty of sub-barrier tunneling. In Fig. 18 we see that a potential well V_{22} is formed in the second channel in the vicinity of the potential barrier in the first channel, and the wave from the first channel is partially transferred to this well. If the energy of the incident wave lies between the two thresholds, the second channel is closed and the wave in it is *trapped* in the well, from which it can decay only into the first channel, although toward both sides. It turns out that such *decay waves traveling to the right have the same amplitude as those reflected from the barrier in the first channel, but opposite phase, so that they cancel.* This is reminiscent of the source of the resonance tunneling effect proposed in Ref. 2 (by one of the authors, B.Z.). However, there this sort of cancellation of the reflected and decay waves occurred only at a single value of the energy (or at a finite number of energies) in the continuum (on a set of measure zero in the spectral continuum). Why is this possible at all continuum energies in the two-channel case? The additional degree of freedom arises because of the nondiagonal elements of the interaction matrix (the continuous degree of freedom in the choice of V_{12}). It is also useful to compare this barrier "avoidance" effect (see Fig. 20) with the similar effect in wave motion in higher-order equations[6] (which we have termed "generalized Schrödinger equations"), except

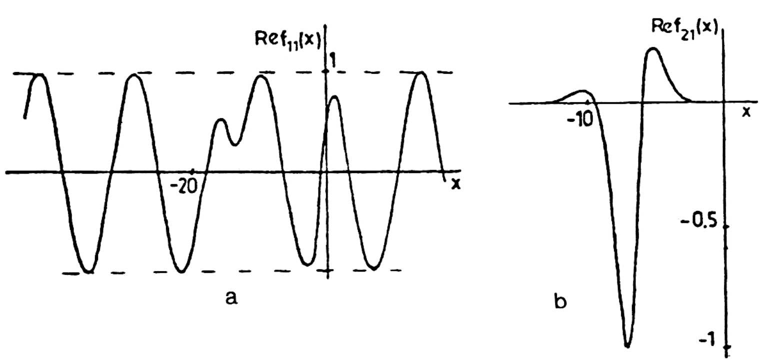

FIG. 19. Channel functions (only their real component is shown) for the energy $E=0.2$ between thresholds. In the open first (a) channel with two sides outside the interaction region the solution is an unperturbed sinusoidal function with unit amplitude. In the closed (b) channel the solution is exponentially damped on both sides.

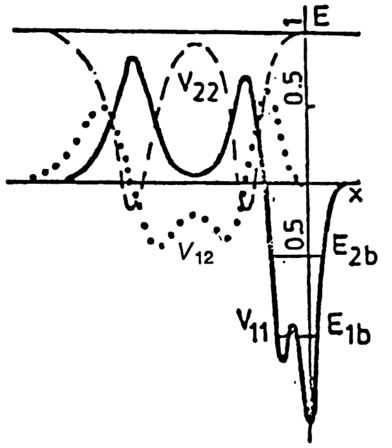

FIG. 21. Transparent interaction potential matrix with two levels $E_1=-0.5$, $M_{11}=1$, $M_{12}=0.01$ and $E_2=-1$, $M_{21}=1.2$, $M_{22}=0.001$. In the first channel in V_{11}, shown by the solid line, two barriers are formed, while in the second channel in $V_{22}+\varepsilon_2$ (dashed line) two "compensating" wells are formed at the same locations. The dotted line shows the nondiagonal matrix element V_{12} which couples the channels and is equal to zero at the barrier maxima.

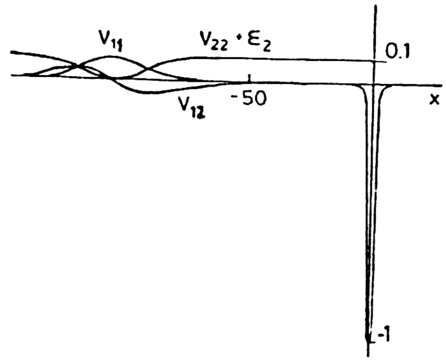

FIG. 23. The same as in Fig. 18 but with the difference between the thresholds for the two channels decreased by a factor of 10. The barrier in V_{11} is shifted to the left as the threshold difference decreases, and all the elements of the interaction matrix become wider.

that there the avoidance path was not a different channel, but a different branch of the spectrum.

Having become acquainted with the one-level interaction matrix, it is not difficult to find the "components" of the more complicated two-level matrix $V_{ij}(x)$ (see Fig. 21), shown in Fig. 18.

By changing the level locations E_i, which control the well depths and barrier heights, and the normalizations M_i of the bound states, which fix the well and barrier locations on the x axis, we obtain a large class of multichannel reflectionless models.

It would be instructive to construct potential filters giving nearly 100-percent reflection at certain selected values of the energy or on certain finite segments of the continuum and nearly 100-percent transmission on the others.

The phenomenon of multichannel transmission is ensured by three features of the interaction matrix. The barriers in its matrix elements give strong reflection, but sub-barrier tunneling is supplemented by barrier avoidance via other channels, where potential wells are located instead of barriers, and the decay waves from these wells in the entrance channel cancel with the waves reflected from the barriers.

The models studied, although they do not correspond to real objects in the quantum world, enrich our experience in understanding qualitatively new mechanisms of enhancing the transmission of composite systems: waves use the possibility of making a channel-to-channel transition to avoid barriers and interchannel interference to suppress reflection.

For completeness, in Figs. 22 and 23 we again give the interaction matrix with the difference between the channel thresholds increased and decreased by a factor of ten, and in Fig. 24 we give the three-channel transparent interaction matrix. In the latter case the wells and barriers in neighboring channels are for some reason slightly shifted relative to each other (cf. Fig. 18). The forms of all the nondiagonal elements are quite similar.

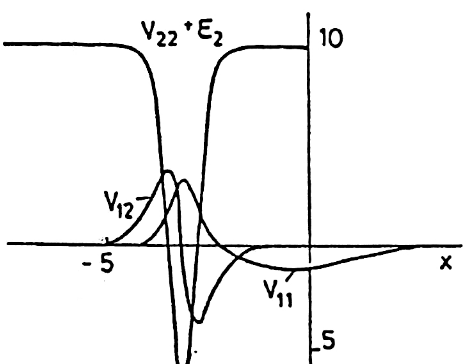

FIG. 22. The same as in Fig. 18 but with the difference between the thresholds for the two channels increased by a factor of 10.

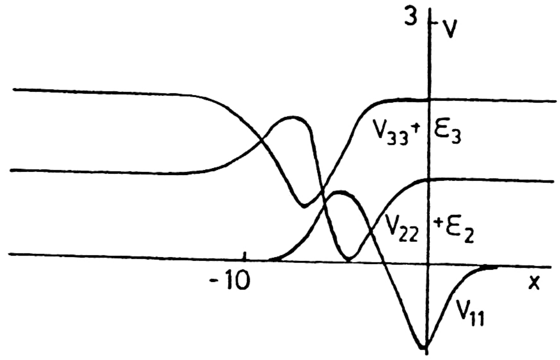

FIG. 24. Diagonal elements of the reflectionless interaction matrix for a three-channel system with bound state $E=-0.5$ (with partial normalization constants $M_1=1$, $M_2=0.1$, and $M_3=0.01$). The channel thresholds are 0, 1, and 2. In order to see them more easily, the diagonal elements are measured from the corresponding thresholds (the matrix becomes zero in the asymptotic limit). Cf. the two-channel matrices in Figs. 18, 22, and 23.

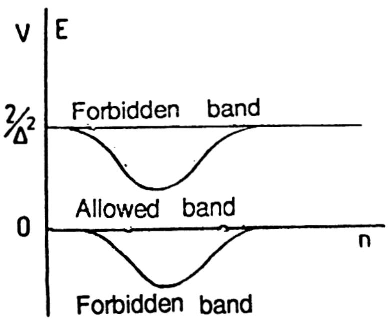

FIG. 25. The local potential $\nu(x)$ bends the allowed band in the (E,n) plane without changing its width; cf. the action of the minimally nonlocal component of the potential controlling the band width, as shown in Fig. 26. The discrete values of the functions in Figs. 25–28 are connected for clarity by the solid lines.

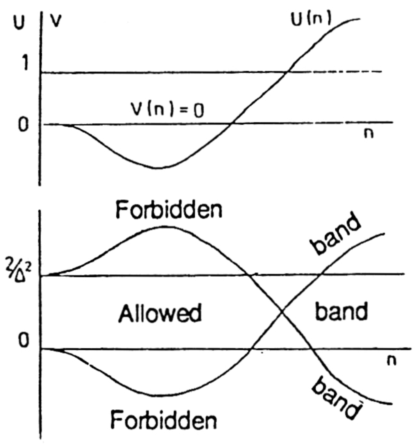

FIG. 26. Change of the width of the allowed band as a function of n, controlled by the potential $u(n)$ determining the effect of adjacent points on each other (minimal nonlocality).

5. TRANSPARENT POTENTIAL PERTURBATIONS ON LATTICES AND IN A PERIODIC FIELD

Elements of the qualitative theory of wave motion along channels

The special feature of wave propagation in a discrete variable n is manifested in the impossibility of constructing a local potential $\nu(n)$ which is transparent at all energies in an allowed band. Any local potential well which bends the allowed band in the E,n plane (see Fig. 25) causes its upper boundary to drop, i.e., it gives rise to a barrier hanging down from above which must produce reflection. A minimal nonlocality [potentials $u(n)$ on the next-to-principal diagonals of the matrix finite-difference Hamiltonian] allows the interfering protrusions of the upper forbidden zone to be corrected. In addition, potentials $\nu(n)$ and $u(n)$ which are soliton-like in shape turn out to be reflectionless. In a periodic field the role of the minimally nonlocal perturbation is played by the matching conditions for the wave functions of neighboring intervals (periods).

5.1. Introductory remarks

In parallel with ordinary quantum mechanics with continuous coordinates, the discrete version describing wave motion along lattices is being developed. The finite-difference Schrödinger equation is used to describe waves in crystals and can serve as a model for studying wave propagation along lattices of discrete variables, for example, variables numbering coupled channels or mixed configurations.[1]

Exactly solvable models aid in the deeper understanding of the features of quantum physics. A large number of extensive classes of such models in ordinary quantum mechanics have been obtained by using supersymmetry methods (Refs. 3, 4, and 42).

It is possible[12,52] to construct a potential which is transparent for waves on a lattice at any value of the energy in the continuum (in an allowed band), using the approach of supersymmetric quantum mechanics (Hamiltonian factorization).

On the one hand, by analogy with the continuous case, one might expect a transparent discrete potential $\nu(n)$ to have a soliton-like shape. However, the local potential bends a band of allowed motion (in the E,n plane; see Fig. 25) without changing its width, so that an attractive well bends the upper boundary of an allowed band downward, which creates an effective barrier from the upper forbidden band hanging down into the allowed band. It would seem that this makes the construction of a transparent system impossible. Help in finding a solution to this problem was obtained from the intuition acquired in Refs. 1, 58, and 59 on spectrum control in discrete quantum mechanics. It is necessary to introduce, in addition to $\nu(n)$, a coupling of functions at neighboring discrete points, i.e., a minimal nonlocality (tridiagonality) of the potential. The values of the potential on the diagonals adjacent to the principal one will be denoted by $u(n)$:

$$-[\psi(n+1)-2\psi(n)+\psi(n-1)]/\Delta^2 + u(n+1)\psi(n+1)+\nu(n)\psi(n) + u(n)\psi(n-1)=E\psi(n), \qquad (18)$$

where δ is the step of the difference differentiation. Using these nondiagonal potential terms $u(n)$, it is possible to control the width of the allowed band[1,58] (see Fig. 26).

This is similar to the introduction of the variable step $\Delta/[1-n(n)]$. In particular, by introducing a soliton-like well $\nu(n)$ it is possible to remove a barrier hanging down from above, thereby flattening the upper boundary of the forbidden zone. As in the continuous case, the family of such transparent potentials depends on $2N$ continuous parameters: the locations of the energy levels of the N bound states and the corresponding normalization constants determining the location of the localization of the bound states on the n axis. It is also possible to construct transparent potentials with bound states above (!) the continuum or with discrete energy levels both below and above an allowed band.

After the transparent perturbations on lattices were understood, the question arose of explaining the reflectionless perturbations $\Delta\nu(x)$ of a periodic field. The approach of the inverse problem gave local perturbations $\Delta\nu(x)$, and the analogy with the discrete case suggested the need for nonlocal additions in order to cancel the effective barriers from the bendings of allowed bands. The way out of the intuitive impasse was that the role of the "effective minimal nonlocality" of the interaction is here played by the ("tridiagonal") couplings of the perturbed solutions on segments of neighboring periods of the unperturbed problem.

5.2. Hamiltonian factorization

Let us construct a transparent potential with a single bound state at a negative energy λ from the functions $\exp(\pm\kappa n)$ satisfying the free (potential-less) equation at the same energy. We shall choose the difference Hamiltonian H_- in the form of a product of conjugate first-order difference operators A^+ and A^- (see Ref. 42):

$$A^-\psi(n)=s(n)\psi(n-1)+r(n)\psi(n), \tag{19}$$

$$A^+\psi(n)=s(n+1)\psi(n+1)+r(n)\psi(n), \tag{20}$$

plus a constant equal to the fixed value of the energy $\lambda=2-1$ $\cosh(\kappa)$ at which we want to produce the bound state:

$$H_-=A^+A^-+\lambda. \tag{21}$$

We take as the solution of the free Schrödinger difference equation [Eq. (18) with $\nu(n)=u(n)=0$] at negative energy λ the linear combination

$$\psi(n)= \exp(\kappa n)+c\ \exp(-\kappa n). \tag{22}$$

From the condition that the coefficients in the free Schrödinger difference equation at the points $n\pm1$ be equal to -1 we have

$$r(n)=-s^{-1}(n), \tag{23}$$

and from (23) and the choice (see Ref. 11)

$$A^-\psi(n)=0 \tag{24}$$

we find

$$r(n)=\sqrt{\frac{\exp(-\kappa(n-1))+c\ \exp(\kappa(n-1))}{\exp(-\kappa n)+c\ \exp(\kappa(n))}}, \tag{25}$$

$$s(n)=-\sqrt{\frac{\exp\ (-\kappa(n))+c\ \exp(\kappa n)}{\exp\ (-\kappa(n-1))+c\ \exp(\kappa(n-1))}}. \tag{26}$$

From the operators (20) we construct the new Hamiltonian

$$H_+=A^-A^++\lambda, \tag{27}$$

which is the supersymmetric partner of H_-, in which the nonzero reflectionless (minimally nonlocal—tridiagonal) potential

$$\nu(n)=s^2(n)+r^2(n) =\frac{\exp(\kappa(n-1))+c\ \exp(\kappa(n-1))}{\exp\ (\kappa n)+c\ \exp(\kappa(n-1))} +\frac{\exp(\kappa(n-1))+c\ \exp(\kappa(n-1))}{\exp(\kappa n)+c\ \exp(\kappa(n-1))}, \tag{28}$$

$$u(n)=1+\frac{\exp(\kappa(n))\ \exp(\kappa n)}{\exp(\kappa(n-1))+c\ \exp(\kappa(n-1))} \times\frac{\exp(\kappa(n-1))+c\ \exp(\kappa(n-1))}{\exp(\kappa n)+c\ \exp(\kappa(n-1))} \tag{29}$$

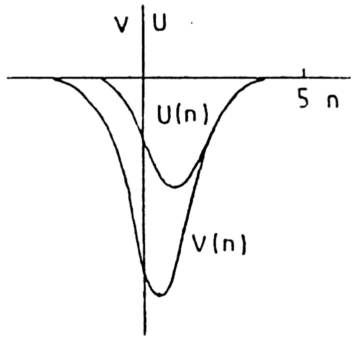

FIG. 27. Soliton-like shape of the potentials $\nu(n)$ and $u(n)$ of a reflectionless difference Hamiltonian. The potential $u(n)$ is needed to "correct" the bending of the forbidden band down into the allowed band caused by the potential $\nu(n)$.

is the difference analog of the soliton potential. The shapes of these potentials $\nu(n)$, $u(n)$ are shown in Fig. 27. The solutions of the equation with the Hamiltonian H_+ are obtained from the free-motion solutions by acting with the operator A_- [see Fig. 28 for $\psi(n)$]. In order to generate a bound state above an allowed band we use the solution

$$\psi(n)=(-1)^n\exp(\kappa n)+c(-1)^n\exp(-\kappa n). \tag{30}$$

The corresponding potentials differ from those shown in Fig. 27 only by the sign of $\nu(n)$, and $u(n)$ is not changed at all, as we could have predicted intuitively (cf. the algorithms for controlling lattice systems in Ref. 1). The wave function $\psi(n)$ of the bound state above the continuum changes sign at each neighboring point, and its modulus is equal to that of the corresponding wave function of the bound state below the continuum.

6. COMMENTARY ON THE LITERATURE

The book by one of the authors (B.Z.) *Lessons in Quantum Intuition* (Ref. 8) is in the process of being printed. A copy on diskette of its continually improving LaTex version, which will coexist with the hard copy, can be obtained from the author.

Rosner[16] has suggested that periodic structures with band spectra be approximated by reflectionless potentials.

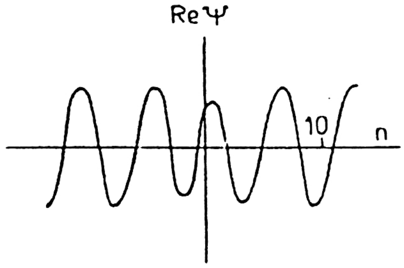

FIG. 28. Scattering and bound-state wave functions in the field of a reflectionless potential.

In Ref. 35 Spiridonov studied exactly solvable models of q-deformed quantum mechanics. Dilatation of the spectrum $E^+ = q^2 E^-$ is obtained in a special case. Further parallels between the method of the inverse problem and the supersymmetry method have become clearer (Refs. 3 and 4). Kay and Moses[60] have generalized the formalism of the inverse problem to the multidimensional case, avoiding the difficulty of the nonmatching of the number of variables on which the scattering and potential data depend by the introduction of a nonlocality in the interaction in the angles, $V(r,\theta,\phi,\theta',\phi')$. The same was achieved in Ref. 2 by using the multichannel formalism. Here $V_{lml'm'}(r)$ depends on four discrete and one continuous variable, which corresponds to nonlocality of the forces, for example, in the angles.

The inverse problem for eigenvalues with discontinuous eigenfunctions was studied in Ref. 26 (1984). The discontinuities of the functions, their positions, and one boundary condition can be reconstructed if the potential is known on half of the interval and if one boundary condition and the eigenvalues are known. The case of symmetric potentials is discussed in Ref. 26.

The inverse problem is uniquely solvable on the entire axis from the reflection coefficients, the bound-state spectrum, and the normalizations only for potentials falling off faster than x^{-2} The loss of uniqueness is demonstrated in Ref. 18.

Quasi-exactly solvable models have been studied in Refs. 27, 30, 32, and 33. So far their relation to exactly solvable models remains unclear.

Sum rules are discussed in several studies.[37]

7. CONCLUSION

The approach of the inverse problem together with the qualitative theory of the control of spectra, scattering, and decays has fundamentally renewed quantum theory. The results obtained here will help in further study of the wave microworld and in the wide variety of applications of it, and they should be introduced into quantum-mechanics courses as quickly and as broadly as possible (in order to provide each physicist with quantum intuition).

Further progress in this science will occur in the development of algorithms for multichannel control. Ideally, it is desirable to isolate the independent parameters (spectral, scattering, and decay parameters), by varying which it is possible to obtain quantum pictures of local potentials. And it would be good to learn how to extract the physical meaning (simple relations between the interaction and observables such as those shown above) from the latter.

The authors would like to thank the Russian Fund for Fundamental Research for support and their coworkers at the JINR Theoretical Physics Laboratory for their stimulating interest in the study.

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Translated by Patricia A. Millard